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SHORTWAVE BIRFURCATION IN A MODEL OF A SEISMICALLY ACTIVE MEDIUM AND DOMINANT FREQUENCIES†

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The evolution equations for nonlinear seismic waves possessing a bounded range of frequencies with increasing amplitudes are analysed. It is shown from the evolution equations that the momentum of the system is conserved, and properties of the energy functional are investigated. The spatial period of the mode with the greatest amplification of the initial perturbation is studied. Conservation of convective nonlinearity leads to a stable stationary structure travelling with the velocity of the nonlinear seismic waves.

1. A GENERALIZED model of a visco-elastic body with internal oscillators was proposed [1, 2] for the mathematical study of nonlinear seismic waves. For weak one-dimensional plane longitudinal waves it reduced to a generalized Burgers–Korteweg De Bries equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \sum_{n=2}^{N}(-1)^n A_n \frac{\partial^n v}{\partial x^n} \tag{1.1}$$

where v is the velocity of the displacement and the A_n are positive numbers. Equation (1.1) was obtained by a perturbation method. This equation is general because it was the case $N = 6$ that was considered. Furthermore, the coefficients A_n were chosen so that there existed a range of oscillation

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frequencies whose amplitude increased with time. It was supposed that these frequencies play a dominant role in seismic waves in actual geomaterials.

We will begin our discussion with some properties of the equation

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{6} (-1)^n A_n \frac{\partial^n v}{\partial x^n} \tag{1.2}$$

to which Eq. (1.1) reduces after linearization. This was used when investigating an early stage of the evolution of the dominant role. Equation (1.2) will be considered either on a line with attenuation conditions for v and all necessary derivatives as $|x| \to \infty$, or in the domain $(0, L)$ with periodic boundary conditions, which is equivalent to setting an initial condition in the form of a spatially periodic function. The choice of this class of problems to be investigated is connected with the properties of the procedure for obtaining (1.1), in which there was a transition to comoving coordinates and the "unbounded" form of the solution domain was implicitly used.

It is clear that both in the linear and nonlinear versions the total momentum

$$M = \int v \, dx \tag{1.3}$$

of the domain is conserved over time.

The behaviour of the quantity

$$E = \int v^2 \, dx \tag{1.4}$$

which is interpreted as the total energy of the medium, is more complicated. To elucidate the time-dependence of (1.4) we consider solutions of Eq. (1.2) for the problem with attenuation conditions at infinity. Changing to Fourier-components v_q such that

$$v = \int_{-\infty}^{\infty} e^{iqx} v_q dq$$

Equation (1.2) reduces to the following ordinary differential equation:

$$v_q^{\cdot} = P(iq) v_q \qquad \left(P(iq) = \sum_{n=1}^{6} (-iq)^n A_n\right) \tag{1.5}$$

From (1.5) we find

$$v(x, t) = \int_{-\infty}^{\infty} C(q) \exp[P(iq)t + iqx] \, dq, \quad C(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x, 0) e^{-iqx} dx$$

Together with (1.5) we consider the conjugate equation

$$v_{-q}^{\cdot} = P(-iq) v_{-q} \tag{1.6}$$

We multiply (1.5) by v_{-q} and (1.6) by v_q and add the results. Then

$$\begin{gathered} d(v_q v_{-q}) / dt = Q(q)(v_q v_{-q}) \\ Q(q) = P(iq) + P(-iq) = 2q^2(-A_6 q^4 + A_4 q^2 - A_2) \end{gathered} \tag{1.7}$$

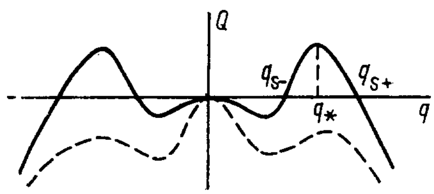

FIG. 1.

Two possibilities follow from this (Fig. 1): either the amplitude of all the Fourier components diminishes with time (the broken curve), or we have a range of wave numbers in which the amplitude of the Fourier components increases. It is also clear that the change in the amplitudes is only governed by the even (dissipative) terms of Eq. (1.2).

By Parseval's equality the energy of the system is equal to

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} (v_q v_{-q})\, dq \tag{1.8}$$

We further obtain

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} | C(q) |^2 \exp (Q(q)\, t)\, dq \tag{1.9}$$

The most important case is when $Q(q)$ has zeroes for nonzero values of q, (the solid curve in Fig. 1). Here the asymptotic behaviour of E for large values of t is given by Laplace's method:

$$E = \frac{2}{\sqrt{-2\pi Q''(q_*)\, t}} |C(q_*)|^2 \exp (Q(q_*)\, t) \tag{1.10}$$

The quantity q_* is given by the vanishing of $Q'(q)$:

$$q_* = \left(\frac{A_4 + (A_4^2 - 3A_2A_6)^{1/2}}{3A_6} \right)^{1/2}$$

It is clear that the energy of the system (the spatially averaged squared amplitude of the seismic waves) increases exponentially with time. Because the total momentum of the medium M is constant, this means that at various points in space the value of v tends either to $+\infty$ or $-\infty$ (Fig. 2).

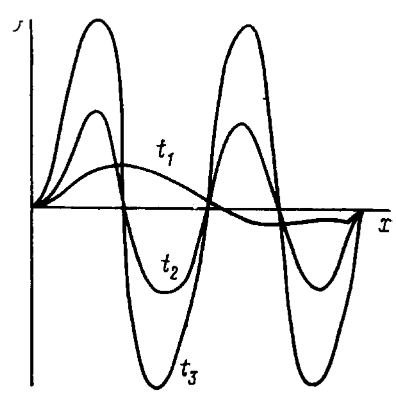

FIG. 2.

2. It was previously noted [1] that this behaviour of E, which is paradoxical at first glance, is associated with its definition: to obtain an expression for the total energy it is necessary to take into account successive approximations in the perturbation method. However, for Eq. (1.2) there nevertheless exists a functional differing from E by the inclusion of derivatives and decreasing with time. Indeed, consider the expression

$$E_* = \int \left(\frac{A_2}{2} v^2 - \frac{A_4}{2} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{A_6}{2} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 \right) dx \tag{2.1}$$

Computing the variational derivative of the functional E_* we obtain

$$\frac{\delta E_*}{\delta v} = A_2 v + A_4 \frac{\partial^2 v}{\partial x^2} + A_6 \frac{\partial^4 v}{\partial x^4} \tag{2.2}$$

We now multiply both sides of Eq. (1.1) by expression (2.2) and integrate over the solution domain. The left-hand side of the resulting expression can be written in the form

$$\int \frac{\delta E_*}{\delta v} \frac{\partial v}{\partial t} dx \equiv \frac{dE_*}{dt} \tag{2.3}$$

and the right-hand side in the form

$$-\int \left(\frac{\partial}{\partial x} \frac{\delta E_*}{\delta v} \right)^2 dx + \int G \frac{\delta E_*}{\delta v} dx \tag{2.4}$$

where G is the sum of terms on the right of (1.2) with odd derivatives with respect to x.

One can verify that the second integral in (2.4) is identically zero. It therefore follows from (2.3) and (2.4) that

$$\frac{dE_*}{dt} = -\int \left(\frac{\partial}{\partial x} \frac{\partial E_*}{\delta v} \right)^2 dx \leqslant 0$$

i.e. E_* is a nonincreasing quantity. As was shown in [1], Eq. (1.2) governs the evolution of transverse waves in a medium with complex visco-elastic rheology. The appearance of higher derivatives in the expression for E_*, which varies "correctly" with time, is directly associated with their presence in the generalized rheological law [1].

Thus the appropriate functional is formed by adding terms to the usual energy E which depend on $\partial v/\partial x$ and $\partial^2 v/\partial x^2$ (cf. [4], where it is shown that the Ginzburg–Landau functional, which depends on the unknown function and its spatial derivatives, plays a similar role in the Cahn–Hilliard equations [5]).

3. The solution of problem (1.2) with periodic initial conditions has the form

$$v = \sum_q C(q) \exp\left[P(iq)t + iqx\right] \tag{3.1}$$
$$C(-q) = \bar{C}(q), \quad q = 2\pi m/L, \quad m = \pm 1, \pm 2 \ldots$$

and the most rapidly increasing modes correspond to the value $q = q_*$, see (1.1). If the initial distribution of v has spatial period L, then the solution possesses interesting properties associated with the discreteness of the set of eigenfunctions of the operator on the right-hand side of (1.2) which were previously considered for the Cahn–Hilliard equation [3]. In particular, we consider

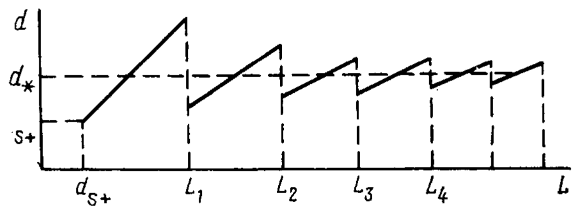

FIG. 3.

how the period of the fastest-growing mode changes as L changes. In the case under investigation the fluctuation amplification coefficient $Q(q)$ is an upwardly convex function in a neighbourhood of the point $q = q_*$. If we change L, then we also change the index of the values $q_m = 2\pi m/L$ nearest to q_*, in the sense of the size of the value of the amplification factor. The condition for changing the leading mode has the form [3]

$$Q\,(2\pi m/L) = Q\,(2\pi\,(m+1)\,/\,L) \tag{3.2}$$

This condition governs the value of L_m at which the mth bifurcation of the period of the fastest growing mode occurs (a change of the kinetically favourable regime).

One can show that for any upwardly convex dependence of the amplification factor and sufficiently large values of m the dependence of the bifurcation parameters has a universal form. To do this we expand the function Q as a series about the value q_* up to squares in the deviation. Using the fact that $Q'(q_*) = 0$ we obtain the asymptotic bifurcation condition

$$(-2\pi m/L + q_*)^2 = (2\pi\,(m+1)\,/\,L - q_*)^2 \tag{3.3}$$

After transformation we obtain

$$L_* = 2\pi q_*^{-1}\,(m + {}^1/_2) = d_*\,(m + {}^1/_2) \tag{3.4}$$

Thus for $L \geqslant d_*$ the period d of the fastest-growing mode is almost identical with d_*. These results can be illustrated by a graph showing the dependence of the period of the fastest-growing mode on L (Fig. 3, see also [3]). It is clear that the dependence consists of linear sections with intervening jumps. The graph has the form of a saw with teeth that decrease as L increases. Because the period of the fastest-growing mode cannot be smaller than the period d_{s+} of the small-amplitude stationary solution of Eq. (1.2), the graph begins at the point d_{s+}, d_{s+}, where $d_{s+} = 2\pi/q_{s+}$ is given by the equation $A_6 q_s^4 - A_4 q_s^2 + A_2 = 0$ and is equal to

$$d_{s+} = 2\pi\left(\frac{2A_6}{A_4 + \sqrt{A_4^2 - 4A_2A_6}}\right)^{1/2} \tag{3.5}$$

The results of this section indicate that for initial perturbations of the medium with characteristic length-scale L much greater than the quantity q_*^{-1}, the spatial period of the fastest-growing mode is almost independent of the form of the initial perturbation. However, if $L \sim q_*^{-1}$, then depending on the variation of L the size of the spatial period of the fastest-growing mode can change quite strongly.

The main difference between Eq. (1.1) and other models in nonlinear dynamics, such as the fourth-order dispersion–dissipation equation with Burgers–Korteweg de Vries nonlinearity [6] or the well-known equation of phase-transition theory [5] is that in our case the left boundary of the spectrum of growing modes $q_{s-} > 0$. (The quantity $q_{s-} = 2\pi/d_{s-}$ is given by formula (3.5) with a

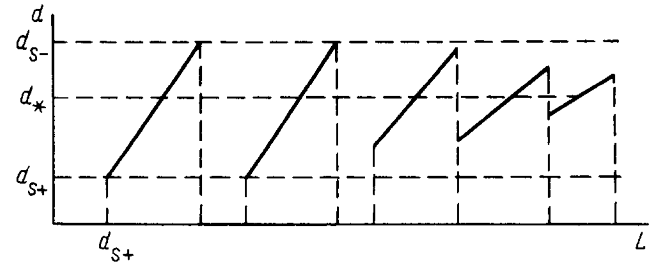

FIG. 4.

minus sign in front of the root in the denominator, see also Fig. 1.) Thus the details of the problem can lead to a change in the dependence $d(L)$ when $L \sim q_*^{-1}$.

In particular, if the coefficients of (1.2) are such that $Q(2q_-)<0$, (i.e. for a sufficiently small region of the spectrum corresponding to the growing modes), then as L increases the first leading mode with $q = 2\pi/L$ leaves the domain (q_{s+}, q_{s-}) before the second leading mode (with $q = 4\pi/L$) has entered it. Hence for one or several leading bifurcations the graph of the function $d(L)$ will differ from Fig. 3: "windows" will exist (see Fig. 4) for which there are no growing modes. the size of L beginning from which the "windows" vanish and the form of $d(L)$ is that shown in Fig. 3 is given by $d_{s-}m$, and the number m is given by the relations $Q(q_{s-}(m+1)/m)>0$, $Q(q_s m/(m-1))<0$. Thus a spatially periodic initial distribution of v might not lead to the evolution of a dominant mode even in the presence of a domain of positive values of $Q(q)$ if the growing modes lie in a small domain of the spectrum and L lies in a window in Fig. 4.

Physically, the exponential growth of the wave amplitude will sooner or later stop, and this is connected mathematically with the "switching-on" of nonlinear effects. One expects that at long times a stationary propagating wave will form.

Indeed, by changing the coefficients of (1.1) the curve $Q(q)$ can be made to touch the q axis at the point q_*: this moment corresponds to a bifurcative creation of a time-periodic solution of (1.1) [7]. A verification that sufficient conditions are satisfied for the periodic orbit that is being formed to be stable for large subcriticality is beyond the scope of this paper. However, the general theory [8–10] developed for uniformly distributed nonlinear systems with instability of the kind shown on Fig. 1, enables us to assert that at least in a weakly subcritical situation a stable stationary structure will form in the form of a propagating wave with a spatial period corresponding to q_*.

4. We will consider in more detail the nonlinear equation (1.1). We will first deal with the nondispersive case, when the right-hand side of (2.1) only contains even derivatives, i.e. the equation has the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = A_2\frac{\partial^2 v}{\partial x^2} + A_4\frac{\partial^4 v}{\partial x^4} + A_6\frac{\partial^6 v}{\partial x^6} \tag{4.1}$$

The trivial solution $v \equiv 0$ becomes unstable under the condition

$$A_4^2 \geqslant 4A_2A_6 \tag{4.2}$$

where the growing perturbations have wave numbers [1] lying in the interval

$$q_{s-}^2 < q < q_{s+}^2, \quad q_{s\pm}^2 = \frac{A_4 \pm \sqrt{A_4^2 - 4A_2A_6}}{2A_6} \tag{4.3}$$

The nonlinear dynamics described by Eq. (4.1) can be investigated analytically near the birfurcation point, i.e. for

$$A_2 = {}^1/_4 A_4{}^2 A_6{}^{-1} (1 - \lambda)^2 \tag{4.4}$$

where λ is a small parameter. The instability appears for infinitely small λ at wave numbers [see 4.3)] $q = q_*$, where

$$q_*{}^2 = {}^1/_2 A_4 A_6{}^{-1} \tag{4.5}$$

It is clear that as $q_{s+} - q_{s-}$ tends to zero (1.11) reduces to (4.5). Following general methods (see e.g. [9, 10]), we shall look for a stationary solution of the nonlinear equation (4.1) of the form

$$v(x) = a_1(q)\cos(qx) + a_2(q)\sin(2qx) + a_3(q)\cos(3qx) + \ldots \tag{4.6}$$

where it is assumed that $|q - q_*| \leqslant \lambda$, $a_1 \sim \lambda^2$, $a_3 \sim \lambda^3$, ..., [where λ is the small parameter introduced in (4.4)]. This assumption will be verified by later calculations.

Substituting (4.6) into (4.1) and equating coefficients of the harmonics, we will first of all express a_2 in terms of a_1 [by equating the coefficients of $\sin(2qx)$]:

$$a_2 = a_1{}^2 (9/32)\, q_*{}^{-1} A_4{}^2 A_6{}^{-1} \tag{4.7}$$

here q_* is given by (4.5). In the derivation of Eq. (4.7) we ignored small terms $\sim\lambda$, λ^2, ... However, when equating the coefficients of $\cos(qx)$ it is necessary to keep small terms $\sim\lambda^3$, (the lower terms $\sim\lambda$ automatically cancel). As a result we obtain

$$a_1\,[{}^1/_2 q_* a_2 - {}^1/_8 A_4{}^3 A_6{}^{-2}\lambda^2 + {}^1/_2 A_4 (q^2 - q_*{}^2)^2] = 0 \tag{4.8}$$

We then substitute expression (4.7) into (4.8) and obtain an equation for the amplitude of the fundamental harmonic $a_1(q)$ [see Eq. (4.6)] which has the trivial solution $a_1 = 0$ and the nontrivial solution

$$a_1{}^2 = (32/9)\, A_6 A_4{}^{-1}\, [({}^1/_2 A_4 A_6{}^{-1})^2 \lambda^2 - (q^2 - q_*{}^2)^2] \tag{4.9}$$

Thus nontrivial solutions of (4.6), (4.7) and (4.9) exists in the wave-number domain

$$(q^2 - q_*{}^2)^2 < ({}^1/_2 A_4 A_6{}^{-1})^2 \lambda^2 \tag{4.10}$$

for a given small subcriticality parameter λ^2.

We note that this is the same domain in which, according to Eqs (4.3)and (4.4), the trivial solution is unstable. The amplitude reaches its maximum value

$$(a_1{}^2)_{\max} = {}^8/_9 A_4 A_6{}^{-1} \lambda^2 \tag{4.11}$$

when $q^2 = q_*{}^2$.

The question of the stability of solutions with amplitudes (4.6)–(4.9) is of interest. In view of the importance of this problem we shall here give a brief account of the main results in general form. To do this we present the general nonstationary solution in the following form:

$$v(x, t) = U_0(x, t) + U_1(x, t)\, e^{ixq_*} + \overline{U}_1(x, t)\, e^{-ixq_*} + U_2(x, t)\, e^{2ixq_*} + \\ + \overline{U}_2(x, t)\, e^{-2ixq_*} \tag{4.12}$$

where U_0 is the slowly-varying real amplitude of the zeroth harmonic, U_1 and U_2 are slowly-varying complex amplitudes of the first and second harmonics, and q_* is taken to be unity. Substituting expansion (4.12) into the original equation of the form (4.1) we obtain a system of generalized Ginzburg–Landau (GL) equations, which after some necessary reduction can be written in the following form:

$$(U_1)_t = \lambda^2 U_1 + 4\,(U_1)_{xx} - \varkappa\,|\,U_1\,|^2\,U_1 - iU_0U_1 \tag{4.13}$$

$$(U_0)_t = \eta^2\,(U_0)_{xx} - (|\,U_1\,|^2)_x \tag{4.14}$$

where λ^2 is the small subcriticality parameter introduced above, and $\varkappa$ and η^2 are arbitrary (not small) parameters.

The presence of the second equation for the slowly relaxing zeroth mode is a major difference between system (4.13), (4.14) and the classical GL equation. The family of stationary periodic solutions has the same form as in the case of the usual GL equation:

$$U_1 = \varkappa^{-1}\,\sqrt{\lambda^2 - 4k^2}e^{ikx} \quad (k^2 \leqslant \lambda^2/4) \tag{4.15}$$

Here $k = q - q_* = q - 1$ is the wave-number separation. However, the stability conditions for solution (4.15) are significantly different from the well-known Eckhaus stability criteria [8] for the solutions of the GL equation

$$|\,k\,| \leqslant \lambda\,/\,\sqrt{12} \tag{4.16}$$

Linearizing Eq. (4.13) and (4.14) about solution (4.15) we find that the stability criteria have the form

$$-{}^1\!/_4\varkappa\,(4 + \eta^2)\,\lambda^2 \leqslant k \leqslant 0 \tag{4.17}$$

Thus the stability domain (4.17) is unusually narrow compared with the classical Eckhaus domain (4.16). Another important difference is that in our case only solutions with $k<0$ can be stable. From the point of view of the original representation (4.12) this means that it is necessary for stability that the total wave number $q = 1 + k$ should be smaller than the critical wave number $q_* = 1$ at which instability appears. To conclude, we note that the above results hold for a problem without boundary conditions, (i.e. specified on the entire $-\infty < x < \infty$ axis).

5. The original equation was obtained [1, 2] in a moving system of coordinates, so that a quiescent system in which a stable stationary structure has appeared corresponds to waves travelling with the velocity c of linear seismic waves. The stability of the stationary structures mean that any "starting" initial perturbation (such as white noise) is transformed into waves with spatial period $2\pi/q_*$ and frequency cq_*. The introduction of odd derivatives into the original equation displaces the dominant frequency from cq_*, and moreover leads to small-amplitude dominant frequency waves propagating in the moving coordinate system [11]. However this "supplementary" velocity should be considerably less than the velocity of sound c. A detailed consideration of the nonlinear equation (1.1) with $A_3 \neq 0$ and $A_5 \neq 0$ should be conducted separately.

In conclusion, we remark that from the formally mathematical point of view the results of this investigation are not restricted to equations of the form (1.1). One could have used any entire function $P(\partial/\partial x)$ as an operator on the right-hand side of (1.1) so long as the function $Q(q) = P(iq) + P(-iq)$ had positive sections. The separation of the growing-mode domains (q_{s-}, q_{s+}) from zero (the shortwave bifurcation condition [8]) was fundamental to the analysis of the stability of weakly-subcritical stationary solutions in the previous section. We note that $N = 6$ is the spatial derivative of minimum order for which shortwave bifurcation is possible for an equation with the operator on the right-hand side depending polynomially on $\partial/\partial x$.

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STABILITY OF THE DISPLACEMENT OF IMMISCIBLE VISCO-ELASTIC LIQUIDS IN A POROUS MEDIUM†

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The stability of the motion of the boundary between two visco-elastic liquids in a porous medium caused by the non-equilibrium of filtration fluxes is investigated. Two stages in the development of the instability are considered: the first is the development of small perturbations on the surface of the initially unperturbed displacement front and the second is the evolution of small perturbations on the surfaces of "fingers" of displacing liquid extruded after the development of the first-stage instability. The theoretical analysis shows that the use of visco-elastic liquids for displacing oil from strata should increase the stability of the displacement process. In the case of visco-elastic liquids with relaxation along the pressure gradient the stability of the displacement is due to stabilization of the actual boundary between the liquids, whereas with relaxation along the flux it is achieved because of the instability of the "fingers" of displacing fluid.

DURING the displacement of liquids of greater viscosity by less viscous liquids or gases there is a viscous instability of the displacement front in a porous medium. Much work (see, e.g. the reviews

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